LAST TIME: FUBINI'S THEOREM

If f is cts on $R:[a,b]\times[c,d]$, then $\int_{y=c}^{d}\int_{x=a}^{b} f(x,y)\,dy = \iint_R f(x,y)\,dA = \int_{x=a}^{b}\int_{y=c}^{d} f(x,y)\,dy\,dx$.

ex: compute $\iint_R \frac{x}{1+xy}\,dA$ on $R=[0,1]\times[0,1]$

sol 1: $\iint_R \frac{x}{1+xy}\,dA = \int_{x=0}^{1}\int_{y=0}^{1}\frac{x}{1+xy}\,dy\,dx$

inner integral: $\int_{y=0}^{1}\frac{x}{1+xy}\,dy$ $\quad\left(\begin{array}{l} u(y)=1+xy \\ du = x\,dy\end{array}\right.$

$$=\int_{y=0}^{1}\frac{1}{1+xy}\cdot x\,dy = \int_{y=0}^{1}\frac{1}{u}\,du = \ln|u|\Big|_{y=0}^{1} = \ln|1+xy|\Big|_{y=0}^{1}$$

$$=\ln(1+x)-\ln(1+0) = \ln(1+x)$$

outer integral: $\int_{x=0}^{1}\ln(1+x)\,dx = \iint_R \frac{x}{1+xy}\,dA$

$$=\left[x\ln(1+x) - \int\frac{x+1-1}{1+x}\,dx\right]_{x=0}^{1} \qquad \left(\begin{array}{ll} u=\ln(1+x) & dv=dx \\ du=\frac{1}{1+x}dx & v=x\end{array}\right.$$

$$=\left[x\ln(1+x) - \int\left(1-\frac{1}{1+x}\right)dx\right]_{x=0}^{1}$$

$$=\left[x\ln(1+x) - (x-\ln(1+x))\right]_{x=0}^{1}$$

$$=(\ln(2)-(1-\ln(2))) - (0-(0-\ln(1))) = 2\ln(2)-1$$

sol 2: inner integral: $\int_{x=0}^{1}\frac{x}{1+xy}\,dx$

- in principle: could try $\left(\begin{array}{l} u(x)=1+xy \rightarrow x=\frac{u-1}{y} \\ du = y\,dx\end{array}\right.$

$$=\int_{x=0}^{1}\frac{\left(\frac{u-1}{y}\right)}{u}\cdot\frac{1}{y}\,du$$

$$=\boxed{\frac{1}{y^2}}\int_{x=0}^{1}\frac{u-1}{u}\,du$$

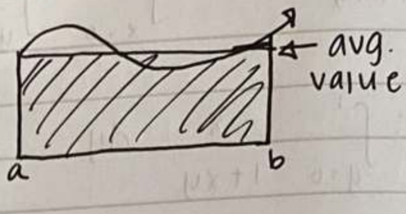
distributes discontinuity at $y=0$

Exercise: compute $\iint_R ye^{-xy}\, dA$ on $R = [0,2] \times [0,3]$. Write out both possible orders of integration.

POINT: Sometimes one order is more computable than another order.

Defn: The average value of function $f(x,y)$ on Region R is

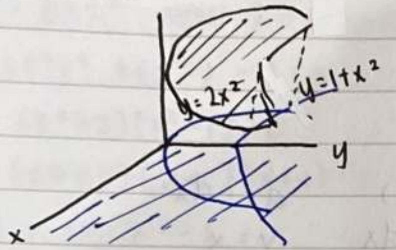
$$\frac{1}{\text{Area}(R)} \iint_R f(x,y)\, dA$$



Goal: Integrate over more than just rectangles...

ex: Compute the (net) volume of the solid bounded by
$y = 2x^2$, $y = 1+x^2$, $z = x+2y$ and $z = 0$

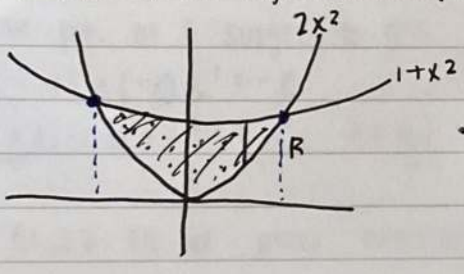
Picture:



Net volume: $\iint_R ((x+2y)-0)\, dA$ over

$R = \{(x,y) : (x,y)$ between $y = 2x^2$ & $y = 1+x^2$

Now a picture of the region in the xy-plane:



- For fixed x, we know $2x^2 \le y \le 1+x^2$
- To find x-bounds, solve $\left(\begin{array}{l} 2x^2 = 1+x^2 \\ \text{iff } x = \pm 1 \end{array}\right)$

$\therefore R = \{(x,y) : -1 \le x \le 1,\ 2x^2 \le y \le 1+x^2\}$

Thus, because our parameterization of R is nice, we can write our double integral as an iterated integral!

Sol: We just saw $R = \{(x,y) : -1 \le x \le 1,\ 2x^2 \le y \le 1+x^2\}$

$$\therefore \iint_R (x+2y)\, dA = \int_{x=-1}^{1} \int_{y=2x^2}^{1+x^2} (x+2y)\, dy\, dx$$

$$= \int_{x=-1}^{1} \left[xy + y^2 \right]_{y=2x^2}^{1+x^2} dx = \int_{x=-1}^{1} \left(x(1+x^2) + (1+x^2)^2 - \left(x(2x^2)) + (2x^2)^2\right)\right) dx$$

$$= \int_{x=-1}^{1} x(1+x^2-2x^2) + ((1+x^2)^2 - (2x^2)^2))\, dx =$$

$\rightarrow$

$= \int_{x=-1}^{1} (x(1-x^2) + (1+x^2+2x^2)(1+x^2-2x^2))\,dx$

$= \int_{x=-1}^{1} (1+x+3x^2)(1-x^2)\,dx$

$= \int_{x=-1}^{1} (1+x+2x^2-x^3-3x^4)\,dx$

$= \left[x + \frac{1}{2}x^2 + \frac{2}{3}x^3 - \frac{1}{4}x^4 - \frac{3}{5}x^5 \right]_{x=-1}^{1}$

$= (1+\frac{1}{2}+\frac{2}{3}-\frac{1}{4}-\frac{3}{5}) - (-1+\frac{1}{2}-\frac{2}{3}-\frac{1}{4}-\frac{3}{5}) = 2\cdot 1 + 0 + 2\cdot\frac{2}{3} - 0 - 2\cdot\frac{3}{5} = 2(1+\frac{2}{5}-\frac{3}{5}) = 2\left(\frac{15+10-9}{15}\right)$

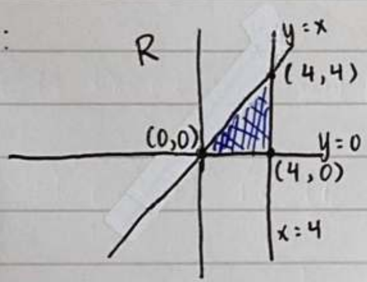
$= \frac{32}{15}$ ☒.

TAKEAWAY: If R is parameterized by something like $R = \{(x,y): c_1 \le x \le c_2,\ g_1(x) \le y \le g_2(x)\}$, then $\iint_R f(x,y)\,dA = \int_{x=c_1}^{c_2} \int_{y=g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$

Similarly, $R = \{(x,y): c_1 \le y \le c_2,\ g_1(y) \le x \le g_2(y)\}$ yields
$\iint_R f(x,y)\,dA = \int_{y=c_1}^{c_2} \int_{x=g_1(y)}^{g_2(y)} f(x,y)\,dx\,dy$

ex: compute $\iint_R y^2 e^{xy}\,dA$ for R bounded by $y=x$, $y=0$, $x=4$

Picture:



$R = \{(x,y): 0 \le y \le 4,\ y \le x \le 4\}$
$R = \{(x,y): 0 \le x \le 4,\ 0 \le y \le x\}$

sol: $\iint_R y^2 e^{xy}\,dA = \underbrace{\int_{y=0}^{4} \int_{x=y}^{4} y^2 e^{xy}\,dx\,dy}_{\text{easier to do...}} = \int_{x=0}^{4} \int_{y=0}^{4} y^2 e^{xy}\,dy\,dx$

$\iint_R y^2 e^{2xy}\,dA = \int_{y=0}^{4} \int_{x=y}^{4} y^2 e^{xy}\,dx\,dy$

inner: $\int_{x=y}^{4} y^2 e^{xy}\,dx = \int_{x=y}^{4} y e^{xy} \underbrace{y\,dx}_{du} = \int_{x=y}^{4} y e^{u}\,du = y[e^u]_{x=y}^{4}$

$\begin{cases} u = xy \\ du = y\,dx \end{cases}$ $= y[e^{xy}]_{x=y}^{4} = y(e^{4y} - e^{y^2}) = ye^{4y} - ye^{y^2}$

$\therefore \iint_R y^2 e^{xy} = \int_{y=0}^{4} (ye^{4y} - ye^{y^2})\,dy$

$= \int_{y=0}^{4} ye^{4y}\,dy - \int_{y=0}^{4} ye^{y^2}\,dy$ IBP: $\begin{cases} u = y & dv = e^{4y}\,dy \\ du = dy & v = \frac{1}{4}e^{4y} \end{cases}$ sub: $\begin{cases} w = y^2 \\ dw = 2y\,dy \end{cases}$ ⟶

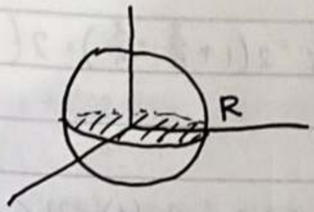
$$= \left[\frac{1}{4} y e^{4y} - \int \frac{1}{4} e^{4y} \, dy - \int e^{x} \, dw \right]_{y=0}^{4} = \left[\frac{1}{4} 4 e^{4y} - \frac{1}{16} e^{4y} - \frac{1}{2} e^{y^2} \right]_{y=0}^{4}$$

$$= \left(e^{16} - \frac{1}{6} e^{16} - \frac{1}{2} e^{16} \right) - \left(0 - \frac{1}{16} - \frac{1}{2} \right) = \frac{1}{16}\left(1 - e^{16}\right) + \frac{1}{2} - \frac{1}{2} e^{2} \quad \blacksquare$$

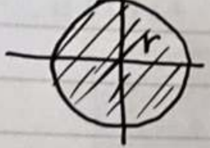
↳ <u>Motivating question</u>: what is the volume of the sphere?

<u>Set up</u>: $S : \{ (x,y,z) : x^2 + y^2 + z^2 = r^2 \}$

$$z = \pm \sqrt{r^2 - x^2 - y^2}$$



• need a nice parametenzation of R



$$R = \{ (x,y) : -r \le x \le r, \ -\sqrt{r^2 - x^2} \le y \le \sqrt{r^2 - x^2} \}$$